FLOW OF COMPRESSIBLE GAS IN A ROUND TUBE OF CONSTANT CROSS SECTION WITH AN IMPERMEABLE ADIABATIC WALL

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Recommendations regarding the calculation of the pressure distribution along a tube and the critical flow rate are derived from an analysis of theoretical and experimental data.

A steady turbulent flow at subsonic velocity is considered below. The integral momentum relationship for an axisymmetric flow of compressible gas, obtained from the Navier–Stokes equations after averaging in accordance with the Reynolds rules and taking into account that the averaged velocity far from the entrance has only one component – along the x axis – and that dp/dr = 0, takes the form

$$\frac{d}{dx}\int\limits_0^1 \rho u^2 \frac{r}{r_0}\, d\,\frac{r}{r_0} = -\frac{1}{2}\cdot\frac{dp}{dx} - \frac{\tau_0}{r_0} + \frac{d}{dx}\int\limits_0^1 \left[\frac{4}{3}\,\mu\,\frac{\partial u}{\partial x} - \overline{(\rho u)'u'}\right]\frac{r}{r_0}\, d\,\frac{r}{r_0}. \quad (1)$$

Here τ_0 is the tangential stress at the wall, which in practical calculations is replaced by a dimensionless drag coefficient

$$\xi = 4\tau_0 \Big/ \int\limits_0^1 \rho u^2 \frac{r}{r_0}\, d\,\frac{r}{r_0}. \quad (2)$$

The quantity $\overline{(\rho u)'u'}$ is the normal turbulent stress due to longitudinal fluctuations of the momentum and velocity. By analogy with the tangential turbulent stress, which is assumed to be proportional to the radial velocity gradient, we can infer that the normal turbulent stress is proportional to the velocity gradient in the direction of the x axis. When this gradient is comparable in value with the radial velocity gradient the last term in Eq. (1) cannot be neglected. As experiments show, a similar situation occurs in the case of a critical flow rate on a small region close to the end of the tube, where the longitudinal velocity gradient increases rapidly, approaching infinity at the end. At flow rates less than the critical rate, and also for the greater part of the tube at critical flow rate, the last term in Eq. (1) can be neglected and the equation then becomes the integral momentum relationship for an axisymmetric boundary layer. The complete system of boundary-layer integral relationships for tube flow without heat transfer has the form:

the continuity equation

$$2\int\limits_0^1 \rho u \frac{r}{r_0}\, d\,\frac{r}{r_0} = \frac{G}{F} = \text{const}; \quad (3)$$

the momentum equation, including (2),

$$\frac{d}{dx}\,2\int\limits_0^1 \rho u^2 \frac{r}{r_0}\, d\,\frac{r}{r_0} = -\frac{dp}{dx} - \frac{\xi}{2r_0}\int\limits_0^1 \rho u^2 \frac{r}{r_0}\, d\,\frac{r}{r_0}; \quad (4)$$

the energy conservation equation with c_p = const

$$2\int\limits_0^1 \rho u\left(T + \frac{u^2}{2c_p}\right)\frac{r}{r_0}\, d\,\frac{r}{r_0} = 2\int\limits_0^1 \rho u T_m \frac{r}{r_0}\, d\,\frac{r}{r_0} = \frac{G}{F}\, T_{m,0} = \text{const}. \quad (5)$$

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Here G/F is the mass flow rate; T_m is the stagnation temperature; $T_{m,0}$ is the same temperature before entrance to the tube. To this system we add the equation of state and for its closure we require to know the local values of the drag coefficient.

In engineering calculations all the quantities are assumed to vary only with length and then the integral relationships are simplified and become one-dimensional equations. In addition, in engineering calculations, according to the results of [15], the drag coefficients can be assumed to be the same as for incompressible fluids. Several investigations have shown, however, that compressibility causes a considerable reduction in ξ in tube flow, particularly at flow velocities close to the velocity of sound. On the other hand, an exactly opposite effect of compressibility on ξ was shown for this flow region in [6]. All these contradictory features are illustrated in Fig. 1.

In the present paper we attempt to discover the causes of the above-mentioned disagreement regarding the effect of compressibility on the drag coefficient in tubes. The limit up to which the use of one-dimensional equations is permissible will also be determined.

The effect of compressibility on drag has been investigated in a number of papers dealing with flow past a flat plate. The results of these investigations are given in [1]. Figure 8 in [2] gives the results of numerous experimental investigations of this problem and compares them with the theoretical calculations of the authors. These investigations show that with increase in M, the ratio of the flow velocity to the velocity of sound, the drag coefficient decreases. At subsonic velocities, however, the effect of compressibility is slight; when M = 1 the drag coefficient for air is 7-12% less than in an incompressible flow. There have been few theoretical studies of the problem of compressible gas flow with a pressure gradient. The problem of tube flow has been dealt with in [3-6]. Without going into a detailed discussion of these studies here, we will merely note that no attempt was made in them to take into account the presence of a longitudinal pressure and velocity gradient. In [6], however, when an attempt was made to determine the special features of the region in tube flow where M increases rapidly, the author based his treatment on boundary-layer equations, which are not applicable to this region. Hence, the results of these studies, like the data on flow past a plate, can hardly be used for a reliable calculation of tube flow, particularly in the near-critical region. It should also be borne in mind that all these studies were based on attempts to use semi-empirical theories derived for incompressible flows and that such attempts "involve a good deal of arbitrariness" [7]. In view of what has been said, experimental investigations acquire great importance and we will now turn to a detailed consideration of these investigations, confining ourselves to those dealing with subsonic flows in thermally insulated tubes.

We note first of all that all the experimental investigations were carried out by identical methods: the gas flow rate, the gas temperature before the entrance to the tube, and the pressure distribution along the tube were measured. The calculations were made from one-dimensional equations. We will consider the validity of using one-dimensional equations instead of integral relations. To do this we transform Eq. (4), introducing the following functions:

$$\varphi_1 = \frac{k-1}{4k} \cdot \frac{G}{F} \cdot \frac{u_{lim}}{p} = \frac{k-1}{4k} \cdot \frac{2}{p} \int_0^1 \rho u u_{lim} \frac{r}{r_0}\, d\frac{r}{r_0}, \tag{6}$$

$$\varphi_2 = \frac{k-1}{4k} \cdot \frac{2}{p} \int_0^1 \rho u^2 \frac{r}{r_0}\, d\frac{r}{r_0}, \tag{7}$$

where $u_{lim} = \sqrt{2c_pT_{m,0}}$ is the limiting velocity and k is the isentropy index. After simple algebra and substituting $d\varphi_2/dx = d\varphi_2/d\varphi_1 \cdot d\varphi_1/dx$, we obtain

$$\frac{\xi}{4r_0}\frac{p}{(-dp/dx)} = 1 + \frac{k-1}{4k\varphi_2} - \frac{\varphi_1}{\varphi_2} \cdot \frac{d\varphi_2}{d\varphi_1}. \tag{8*}$$

For the r_0, φ_1, and $p/(-dp/dx)$ assigned in the experiments the values of ξ will generally be different, depending on the method of calculating φ_2. In the one-dimensional approximation, using the equations of energy and state, we obtain $\varphi = 0.25(\sqrt{1+16\varphi_1^2}-1)$ and (8) becomes

$$\frac{\xi}{4r_0} \cdot \frac{p}{(-dp/dx)} = \frac{k-\sqrt{1+16\varphi_1^2}}{k\sqrt{1+16\varphi_1^2}\,(\sqrt{1+16\varphi_1^2}-1)}. \tag{9}$$

* Incidentally, if we assume that (8) is actually in the critical section, where $dp/dx = -\infty$, then, on equating the right side of (8) to zero and using (6) and (7), we obtain $d2\int \rho u^2(r/r_0)d(r/r_0) = -dp$, i.e., a quasi-isentropic flow, as was obtained in the case of one-dimensional flow [14].

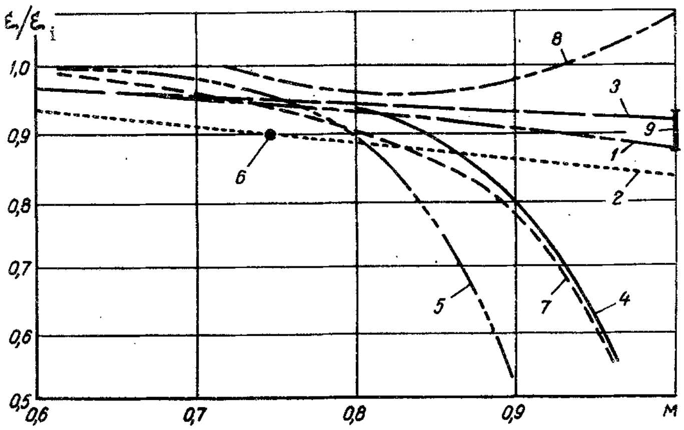

Fig. 1. Ratio of drag coefficient ξ in compressible flow to its value ξ_i in incompressible flow for tube flow of air as a function of the ratio M of flow velocity to sound velocity from the data of different authors: 1) from [3], calculating the turbulent viscosity coefficient by the Nusselt method [19]; 2) according to Zhestkov [see [4], Chap. 6, Eq. (64)]; 3) according to [5]; 4) [9]; 5) [11]; 6) according to [3], calculating the turbulent viscosity coefficient by the Prandtl method; 7) [10]; 8) [6]; 9) according to [2] for flow past a flat plate.

To find φ_2 from the common solution of the integral relations we have to calculate the integrals contained in them. In accordance with the results obtained in [15, 19] we assume a power-law velocity distribution $n = u_1 (1-r/r_0)^{1/n}$. The calculation was conducted for two extreme cases: 1) temperature and, hence, density constant in the cross sections of the flow; 2) stagnation temperature constant over the cross section (and, hence, throughout the flow), i.e., no energy transfer between the stream filaments. In the first case the integrals were calculated in quadratures. We obtained $\varphi_2 = 0.5\, c_1 (\sqrt{1 + 4\varphi_1^2/c_2} - 1)$, where c_1 and c_2 are very weak functions of n. For n varying from 7 to 15, c_1 varies from 0.482 to 0.495 and c_2 from 0.236 to 0.246 (when $n = \infty$, $c_1 = 0.50$ and $c_2 = 0.25$). Substituting their mean values we obtain $\varphi_2 = 0.244 \cdot (\sqrt{1 + 16.28\varphi_1^2} - 1)$ and

$$\frac{\xi}{4r_0} \cdot \frac{p}{(-dp/dx)} = \frac{k - 1{,}0223\sqrt{1 + 16{,}28\varphi_1^2} + 0.0223\,k\sqrt{1 + 16.28\varphi_1^2}}{k\sqrt{1 + 16.28\varphi_1^2}\left(\sqrt{1 + 16.28\varphi_1^2} - 1\right)}. \tag{10}$$

In the second case the substitution $\rho = p/RT_m[1-(u/u_{lim})^2]$ was made and the integrand $[1-(u/u_{lim})^2]^{-1}$ was expanded in a series of powers of $u/u_{lim} = (u_1/u_{lim})(1-r/r_0)^{1/n}$, where u_1 is the velocity on the axis. Integration term by term gives

$$\varphi_1 = \sum_{i=0}^{\infty} n^2 (u_1/u_{lim})^{2i+1} (2i + 1 + n)^{-1} (2i + 1 + 2n)^{-1},$$

$$\varphi_2 = \sum_{i=0}^{\infty} n^2 (u_1/u_{lim})^{2i+2} (2i + 2 + n)^{-1} (2i + 2 + 2n)^{-1}. \tag{11}$$

An investigation showed that we could limit ourselves to three terms of each series. The calculations, made in the range $u_1/u_{lim} = 0\text{–}0.5$ (0.5 is the maximum value of u_1/u_{lim} for subsonic flow and $k = 1.67$) and $n = 7\text{–}15$ showed that in this case also φ_2 depends uniquely on φ_1 with a deviation of not more than 1%. Using a plot of φ_2 against φ_1 we calculated the right side of Eq. (8) and found the values of $(\xi/2r_0) \cdot p/(-dp/dx)$ in the second limiting case.

We found that the drag coefficients calculated from the one-dimensional equations and from the integral relations for $k = 1.3$, 1.4, and 1.67 did not differ from one another by more than 3–4% up to $M = 0.75\text{–}0.80$. Thus, in this region the use of one-dimensional equations is quite permissible.

The region $0.8 \leq M \leq 1$ at subsonic flow velocities in a tube occupies a small relative length $x/r_0 = 6\text{–}8$ close to the exit and is characterized by very large longitudinal velocity gradients. In this region the validity of using boundary-layer relations, particularly the one-dimensional equations, is very doubtful. In addition, the representation of the velocity profile in this region by a power-law formula with a constant power of r is also questionable. It was found in [13] that, in a small region behind the narrow cross section in the nozzle, part of the profile is of a laminar nature. The possibility of laminarization of the flow by large negative pressure gradients has been reported in other investigations.

We now consider the results of the available investigations.

Frossel [15] concluded that the drag coefficients in the case of air flow in tubes were the same as for an incompressible fluid, in complete correspondence with the Nikuradze formula for subsonic and supersonic flow velocities. It should be noted, however, that the graph illustrating this conclusion in [15] contains only one point for each set of conditions. We must infer that these points illustrate only the average values of the drag coefficient over the whole length of the tube.

Only the local values of ξ averaged over the length of the tube were given in [18] and it was shown they were the same as for incompressible flow.

In our investigation [8] with superheated water vapor we showed that up to $M = 0.8$ the compressibility has no effect on the local values of the drag coefficient. No calculations were made for $M > 0.8$, since we found that the error in calculating the drag coefficient due to the error in measurement of the experimental parameters G/F and T_m increases along the tube, particularly strongly in the end region, and reaches an infinitely high value when $M = 1$. To check the reliability of the results obtained up to $M \leq 0.8$ from Eq. (9), we also carried out the reverse calculation by differentiation of the experimental pressure distribution curves. Taking ξ according to Nikuradze and assuming it to be constant over the length since the Reynolds number varies very little with length in adiabatic flow, we calculated the pressure distribution over the length.* The results of the calculations showed that up to cross sections at a distance of $x/r_0 = 8$ from the exit section the measured values were in good agreement with the calculated values.

In an investigation with water vapor [16] only the average drag coefficients over the length were calculated. They were found to be approximately 30% higher than the Nikuradze values. This may be explained in that the tube was composed of four segments and the measured pressure drops included local drag loss due to imperfect butting.

In [17] the critical flow rate was reached, in one of the four experiments conducted with subsonic air flow velocities. The local drag coefficient at $M = 0.84$ (the highest value at which it was calculated) was 3% lower than the Nikuradze value.

In experiments with air [9] the calculations were made up to $M = 0.97$. The authors noted that in the region $0.7 < M < 0.9$ there was a slow reduction, and at $M > 0.9$ a sharp reduction, of the drag coefficient. Using the tables of measured pressure distributions in these experiments we considered them in conjunction with our experiments with flows of argon and carbon dioxide [12]. We found that for gases of different atomicity the effect of compressibility is best allowed for by the number Λ – the ratio of the flow velocity to the maximum velocity in the case of a friction-free flow into a vacuum. We found that when $\Lambda < 0.3$ (this corresponds to $M = 0.826$ for CO_2, 0.71 for air, and 0.582 for argon) we can take $\xi = \text{const}$, and when $\Lambda > 0.3$ the drag coefficients calculated by using Eq. (9) decrease slowly.

In the present investigation we calculated the pressure distribution in the experiments of [9], assuming the drag coefficient constant along the length (as Nikuradze does). The results of these calculations showed that, as in [8], the differences between the calculated and measured pressures were very small over the greater part of the flow. These differences become significant only in the end region of the tube, where $M > 0.8$.

It was noted in [10] for air that in the region up to $M = 0.8$ the drag coefficients are independent of M, but subsequently they decrease sharply. An even more rapid reduction of the drag coefficient with increase

* When $\xi = \text{const}$, Eq. (9) integrates to

$$\frac{k+1}{k}\ln\frac{z}{z_1}+\frac{k-1}{k}\left(\frac{1}{z}-\frac{1}{z_1}\right)=-\xi\frac{x-x_1}{4r_0}, \tag{12}$$

where

$$z \equiv \frac{1+\frac{\varphi_1^2}{2}-\sqrt{1+\varphi_1^2}}{\varphi_1^2/2}.$$

in M in the region close to M = 1 was found in [11]; we cannot analyze this result owing to the absence of experimental points. The variation of the drag coefficient with M for M > 0.8, according to [9-11], is shown in Fig. 1.

Finally, Depassel [19] illustrated the measured pressure distribution and the distribution calculated on the assumption that ξ = const in an experiment with critical air flow rate and found good agreement between them up to the cross section at a distance of $x/r_0 = 9.8$ from the end, where M = 0.83 according to the measured pressure and M = 0.817 according to the calculated pressure.

In the light of the above, we can assert that for di- and triatomic gases at least we can neglect the effect of compressibility on the drag coefficient until the flow velocity exceeds 75-80% of the velocity of sound and can calculate tube flow from Eq. (12).

Calculations for the end region of flow in a tube at critical flow rate cannot be made from the integral boundary-layer relations, far less from one-dimensional equations, owing to the large pressure gradients. The sharp reduction in drag coefficient with increase in M near M = 1, found in the cited investigations, is most probably a result of using one-dimensional equations for the calculation, as was indicated in [6]. This emerges from the following considerations.

In the cross section where M = 1 the common solution of the equations of continuity, energy, and state with the substitution of the velocity of sound for the flow velocity gives a single-valued relationship between the pressure p_{cr} in this section and the critical flow rate G_{cr} in the form $p_{cr} = (G_{cr}/F)\sqrt{2RT_m/k(k-1)}$. In addition, all the experiments show that at the critical flow rate the pressure in the exit section of the tube is less than p_{cr}. Hence, in the section where calculations give M = 1 the pressure gradient has a finite value, since after this section the pressure continues to fall. Substituting the expression for p_{cr} in φ we obtain zero on the right side of Eq. (9) and, consequently, $\xi = 0$. Hence, calculation from one-dimensional equations and from the experimental pressure distribution curve will give $\xi = 0$ when M = 1 and, consequently, a rapid reduction of ξ in some region close to M = 1.

The absence of information about the structure of a turbulent flow in the region close to M = 1, particularly about the effect of turbulent normal stresses, does not allow us at present to use Eq. (1) and the corresponding energy equation.

The critical flow rate and the pressure p_{min} established at the end of the tube in this case can be calculated from the empirical relationships obtained in [15] for fairly long tubes ($l/r_0 > 72$):

$$p_{min}/p_k = 0.911\left(\frac{2}{k+1}\right)^{k/(k-1)} \frac{G_{cr}}{G_{cr,0}}, \tag{13}$$

$$G_{cr}/G_{cr,0} = 0.916^{\left(\frac{l/r_0}{20}\right)^{0.619}}, \tag{14}$$

where $G_{cr,0}$ is the critical flow rate in a loss-free flow:

$$G_{cr,0} = F\,[2/(k+1)]^{1/(k-1)} \sqrt{\frac{2k}{k+1}\,p_k\rho_k}, \tag{15}$$

p_k and ρ_k are the pressure and density in the reservoir before the tube, and p_{min} is the pressure at the end of the tube at critical flow rate, denoted by G_{cr}.

One must bear in mind that in the experiments in [15] the entrance to the experimental tube was rounded. Hence, in the case of a sharp entrance the value p_{min} should be referred to the value of p_k with the pressure loss due to the entrance resistance subtracted.

LITERATURE CITED

1. D. B. Spalding and S. W. Chi, in: Mechanics [Russian translation], No. 6 (1964).
2. S. S. Kutateladze and A. I. Leont'ev, The Turbulent Boundary Layer of a Compressible Gas [in Russian], Izd. SO AN SSSR (1962).
3. M. F. Shirokov, Physical Bases of Gas Dynamics [in Russian], GITTL, Moscow (1957).
4. G N. Abramovich, Applied Gas Dynamics [in Russian], GITTL, Moscow (1953).
5. F. S. Voronin, Inzh-Fiz. Zh., 2, No. 11 (1959).
6. M. M. Nazarchuk, Flow of Gas in Channels in Presence of Heat Transfer [in Russian], Izd. AN UkrSSR, Kiev (1963).

7. H. Schlichting, Boundary Layer Theory [Russian translation], Nauka, Moscow (1969).
8. V. L. Lel'chuk, Zh. Tekh. Fiz., 7, Nos. 18-19 (1937).
9. A. F. Gandel'sman et al., Zh. Tekh. Fiz., 24, No. 12 (1954); A. F. Gandel'sman, Candidate's Dissertation [in Russian], Moscow (1957).
10. B. S. Petukhov et al., Teploénergetika, No. 3 (1957).
11. S. A. Labinov, Izv. VUZ, Énergetika, No. 6 (1962).
12. V. L. Lel'chuk and G. I. Elfimov, in: Heat and Mass Transfer [in Russian], Vol. 1, Énergiya, Moscow (1968), p. 479.
13. A. A. Gukhman et al., in: Heat and Mass Transfer [in Russian], Vol. 1, Énergiya, Moscow (1968), p. 812.
14. I. I. Novikov, Zh. Tekh. Fiz., 19, No. 6 (1948).
15. W. Frössel, Forsch. Ing. Wessen, 7, No. 2 (1936).
16. J. H. Keenan, J. Appl. Mech., 6, No. 1 (1939).
17. J. H. Keenan and E. P. Neumann, J. Appl. Mech., 13, No. 2 (1946).
18. J. E. Bialokoz and O. A. Saunders, Heat Transfer in Pipe Flow at High Speeds, Inst. Mech. Engrs., London (1955).
19. R. Depassel, Ecoulement de l'Air a Grande Vitesse dans un Tuyau de Section Circulaire, Publications Scientifiques et Techniques du Ministere de l'Air, Paris (1960).
20. W. Nusselt, Tech. Mech. Thermodyn., 1, 278 (1930).